A SHORT PROOF OF KONTSEVICH CLUSTER CONJECTURE

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The aim of this note is to give an elementary proof of the following Kontsevich conjecture.

Recall that the *Kontsevich map* K_r, $r \in \mathbb{Z}_{>0}$ is the following (birational) automorphism of a noncommutative plane:

$$K_r : (x, y) \mapsto (xyx^{-1}, (1 + y^r)x^{-1}) \ ,$$

Conjecture 1. *(M. Kontsevich) For any $r_1, r_2 \in \mathbb{Z}_{>0}$ all iterations $\underbrace{\cdots K_{r_1} K_{r_2} K_{r_1}}_{k}(x, y)$, $k \geq 1$ are given by noncommutative Laurent polynomials in x and y.*

The Kontsevich conjecture was first proved for $r_1 = r_2 = 2$ by A. Usnich in [5] and was later settled by A. Usnich in [6] in greater generality when $r_1 = r_2 = r$ (with $1 + y^r$ replaced by any monic palindromic polynomial $H(y)$) by means of derived categories. Independently, Conjecture 1 was verified for $(r_1, r_2) \in \{(2, 2), (4, 1), (1, 4)\}$ in [3] along with the positivity conjecture: for $(r_1, r_2) \in \{(2, 2), (4, 1), (1, 4)\}$ all noncommutative Laurent polynomials in question have nonnegative integer coefficients.

Our goal is to give a short proof of Conjecture 1.

Theorem 2. *For any $r_1, r_2 \in \mathbb{Z}_{>0}$ all iterations $\underbrace{\cdots K_{r_1} K_{r_2} K_{r_1}}_{k}(x, y)$, $k \geq 1$ are given by noncommutative Laurent polynomials in x and y.*

To present our proof of Theorem 2, we need some notation. Denote

$$(x_k, y_k) := \underbrace{\cdots K_{r_1} K_{r_2} K_{r_1}}_{k}(x, y)$$

and denote $z := [x, y] = xyx^{-1}y^{-1}$. Then it is easy to see by induction that $[x_k, y_k] = [x, y] = z$ for all k. This taken together with the recursion $x_{k+1} = x_k y_k x_k^{-1}$ and $y_{k+1} = (1 + y_k^{r_k})x_k^{-1}$, where

$$r_k = \begin{cases} r_1 & \text{if } k \text{ is odd} \\ r_2 & \text{if } k \text{ is even} \end{cases} \tag{1}$$

gives the following three recursions (they first appeared in [3, Section 2.2])

$$x_{k+1} = zy_k, \ y_{k+1}zy_{k-1} = 1 + y_k^{r_k}, \ y_{k+1}zy_k = y_k y_{k+1} \ .$$

Let $\mathcal{F}_2 = \mathbb{Q}\langle y_1^{\pm 1}, y_2^{\pm 1}\rangle$ be the group algebra of the free group in 2 generators. It was proved by A.I. Malcev (see e.g., [4, Section 8.7]) that $\mathcal{F}_2$ is a *divisible algebra*, i.e., it embeds in a division ring (we denote the smallest one by $Frac(\mathcal{F}_2)$).

Define elements $y_k \in Frac(\mathcal{F}_2)$, $k \in \mathbb{Z} \setminus \{1, 2\}$ recursively by:

$$y_{k+1} z y_{k-1} = 1 + y_k^{r_k} \ , \tag{2}$$

where $z := [y_2^{-1}, y_1] = y_2^{-1} y_1 y_2 y_1^{-1}$.

Note that $y_0, y_3 \in \mathcal{F}$ and let $\mathcal{A} = \mathcal{A}(r_1, r_2)$ be the subalgebra of $\mathcal{F}$ generated by $y_0, y_1, y_2, y_3, z, z^{-1}$. We will refer to $\mathcal{A}$ as a *(purely) noncommutative cluster algebra* of type (r_1, r_2).

Lemma 3. *The elements $y_k \in Frac(\mathcal{F}_2)$ satisfy for all $k \in \mathbb{Z}$:*

$$y_{k+1} z y_k = y_k y_{k+1} \tag{3}$$

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Proof. Indeed, the (3) is obvious for $k=1$. Let us prove it for $k\ge 1$ by induction. We will use the inductive hypothesis in the form $y_k y_{k-1}^{-1}z^{-1}=y_{k-1}^{-1}y_k$. Indeed, since $y_{k+1}z=(1+y_k)^{r_k}y_{k-1}^{-1}$, we obtain

$$y_{k+1}zy_k-y_ky_{k+1}=(1+y_k)^{r_k}y_{k-1}^{-1}y_k-y_k(1+y_k)^{r_k}y_{k-1}^{-1}z^{-1}$$
$$=(1+y_k)^{r_k}y_{k-1}^{-1}y_k-(1+y_k)^{r_k}y_ky_{k-1}^{-1}z^{-1}=(1+y_k)^{r_k}y_{k-1}^{-1}y_k-(1+y_k)^{r_k}y_{k-1}^{-1}y_k=0$$

by the inductive hypothesis. The relation (3) for $k\le 0$ also follows. □

Thus, based on the above discussion, Theorem 2 directly follows from our main result.

Main Theorem 4. *Each y_k belongs to $\mathcal{A}$, e.g., y_k is a noncommutative Laurent polynomial in y_1,y_2.*

Proof. Denote by $\mathcal{A}_k=\mathcal{A}_k(r_1,r_2)$ the subalgebra of $\mathcal{F}_2$ generated by $y_k,y_{k+1},y_{k+2},y_{k+3},z^{\pm 1}$. It suffices to prove the following result (which is a noncommutative version of [1, Formula (4.12)] and [2, Lemma 5.8]).

Theorem 5. *$\mathcal{A}_k=\mathcal{A}$ for all $k\in\mathbb{Z}$.*

Proof. Since $\mathcal{A}=\mathcal{A}_0$, it suffices to prove that $\mathcal{A}_k=\mathcal{A}_{k+1}$ for $k\in\mathbb{Z}$, i.e., that for all $k\in\mathbb{Z}$ one has

$$y_{k+4}\in\mathcal{A}_k,\ \ y_k\in\mathcal{A}_{k+1} \tag{4}$$

Proposition 6. *For each $n\in\mathbb{Z}$ one has: $y_{k+4}z=zy_k(y_{k+3}z)^{r_{k+1}}-\sum\limits_{j=0}^{r_{k+1}-1}(zy_{k+1})^jz(y_{k+2}z)^{r_k-1}(y_{k+3}z)^j$.*

Proof. For simplicity (and without loss of generality) we assume that $k=0$. We start with the following technical result.

Lemma 7. *For each $m\ge 0$ we have: $y_1^m(y_3z)^m=1+\sum\limits_{k=0}^{m-1}y_1^k(y_2z)^{r_2}(y_3z)^k$.*

Proof. We proceed by induction on m. For $m=0$ the assertion is clear. Assume that $m>0$ and it holds for $m-1$. Let us prove it for m. Note that the (2) and (3) imply that

$$y_{k-1}y_{k+1}z=1+(y_kz)^{r_k} \tag{5}$$

Indeed, using (5), we obtain

$$y_1^m(y_3z)^m=y_1^{m-1}(y_1y_3z)(y_3z)^{m-1}=y_1^{m-1}(1+(y_2z)^{r_2})(y_3z)^{m-1}=y_1^{m-1}(y_2z)^{r_2}(y_3z)^{m-1}+y_1^{m-1}(y_3z)^{m-1}$$
$$=y_1^{m-1}(y_2z)^{r_2}(y_3z)^{m-1}+1+\sum_{k=0}^{m-2}y_1^k(y_2z)^{r_2}(y_3z)^k=1+\sum_{k=0}^{m-1}y_1^k(y_2z)^{r_2}(y_3z)^k\ .$$

The lemma is proved. □

Furthermore, compute:

$$y_4z=y_2^{-1}((y_3z)^{r_1}+1)=y_2^{-1}(y_3z)^{r_1}+y_2^{-1}=(zy_0-y_2^{-1}(y_1)^{r_1})(y_3z)^{r_1}+y_2^{-1}$$
$$=zy_0(y_3z)^{r_1}-y_2^{-1}(y_1^{r_1-1}(y_1y_3z)(y_3z)^{r_1-1}-1)=zy_0(y_3z)^{r_1}-y_2^{-1}(y_1^{r_1-1}(1+(y_2z)^{r_2})(y_3z)^{r_1-1}-1).$$

We have:

$$y_1^{r_1-1}(1+(y_2z)^{r_2})(y_3z)^{r_1-1}-1=y_1^{r_1-1}(y_2z)^{r_2}(y_3z)^{r_1-1}+y_1^{r_1-1}(y_3z)^{r_1-1}-1\ .$$

Using Lemma 7 and taking into account that $y_1^my_2=y_2(zy_1)^{m-1}$ for $m>0$, we obtain:

$$y_1^{r_1-1}(1+(y_2z)^{r_2})(y_3z)^{r_1-1}-1=y_1^{r_1-1}(y_2z)^{r_2}(y_3z)^{r_1-1}+\sum_{k=0}^{r_1-2}y_1^k(y_2z)^{r_2}(y_3z)^k=\sum_{k=0}^{r_1-1}y_1^k(y_2z)^{r_2}(y_3z)^k$$
$$=y_2\sum_{k=0}^{r_1-1}(zy_1)^kz(y_2z)^{r_2-1}(y_3z)^k\ .$$

Therefore, $y_4z=zy_0(y_3z)^{r_1}-\sum\limits_{k=0}^{r_1-1}(zy_1)^kz(y_2z)^{r_2-1}(y_3z)^k$. This proves Proposition 6. □

Proposition 6 gives us the first inclusion (4). Prove second inclusion (4) now. We need the following obvious fact. Let σ be the anti-automorphism of $\mathcal{F}_2$ given by: $\sigma(y_1) = y_2$, $\sigma(y_1) = y_2$ (so that $\sigma(z) = z$).

Lemma 8. *$\sigma(y_k) = y_{3-k}$ for $k \in \mathbb{Z}$, in particular, $\sigma(\mathcal{A}_k(r_1, r_2)) = \mathcal{A}_{-k}(r_2, r_1)$ for $k \in \mathbb{Z}$.*

This immediately implies the second inclusion (4): $y_{1-k} \in \mathcal{A}_{-k}$, $k \in \mathbb{Z}$ and Theorem 5 is proved. □

Therefore, Theorem 4 is proved. □

And, ultimately, Theorem 2 is proved.

Example 9. Let $r_1 = r_2 = 2$. We have: $y_{k+1}zy_{k-1} = y_k^2 + 1$, $y_{k-1}y_{k+1}z = y_kzy_kz + 1$ for all $k \in \mathbb{Z}$. This implies:

$$y_4z = y_2^{-1}(y_3zy_3z + 1) = (zy_0 - y_2^{-1}y_1^2)y_3(zy_3z) + y_2^{-1}$$
$$= zy_0y_3zy_3z - y_2^{-1}(y_1(y_1y_3z)y_3z - 1)$$

Note that $y_1(y_1y_3z)y_3z - 1 = y_1(y_2zy_2z + 1)y_3z - 1 = y_1y_2zy_2zy_3z + y_1y_3z - 1 = y_2zy_1zy_2zy_3z + (y_2z)^2$. Therefore,

$$y_4z = zy_0(y_3z)^2 - (zy_1zy_2zy_3z + zy_2z).$$

The noncommutative cluster algebra $\mathcal{A} = \mathcal{A}(r_1, r_2)$ has a number symmetries in addition to the anti-involution $\sigma : \mathcal{A}(r_1, r_2)\widetilde{\to}\mathcal{A}(r_2, r_1)$ from Lemma 8: the translation $y_k \mapsto y_{k+1}$, $k \in \mathbb{Z}$ defines an isomorphism $\tau : \mathcal{A}(r_1, r_2)\widetilde{\to}\mathcal{A}(r_2, r_1)$, which is an automorphism when $r_1 = r_2$.

We conclude with a brief discussion of the presentation of $\mathcal{A}$.

Proposition 10. *The generators $y_0, y_1, y_2, y_3, z^{\pm 1}$ of $\mathcal{A}$ satisfy (for $i = 0, 1, 2$, $j = 1, 2$):*

$$y_iy_{i+1} = y_{i+1}zy_i, y_{j+1}zy_{j-1} = y_j^{r_j}+1,\ y_{j-1}y_{j+1}z = (y_jz)^{r_j}+1,\ y_3zy_0-zy_0y_3z = y_2^{r_2-1}y_1^{r_1-1}-z(y_1z)^{r_1-1}(y_2z)^{r_2-1}$$

Proof. Only the last relation needs to be proved (the first three relations are (3), (2), and (5) respectively). Indeed, using the available relations in $\mathcal{F}_2$, we obtain:

$$y_0y_3z = ((1+(y_1z)^{r_1})z^{-1}y_2^{-1})(y_1^{-1}(1+(y_2z)^{r_2})) = (1+(y_1z)^{r_1})z^{-1}y_1^{-1}z^{-1}y_2^{-1}(1+(y_2z)^{r_2}) = h_{r_1}(y_1z)h_{r_2}(y_2z)\,,$$

where $h_r(y) = y^{-1} + y^{r-1}$. Similarly,

$$y_3zy_0 = ((1 + y_2^{r_2})y_1^{-1})(z^{-1}y_2^{-1}(1 + y_1^{r_1})) = (1 + y_2^{r_2})y_2^{-1}y_1^{-1}(1 + y_1^{r_1}) = h_{r_2}(y_2)h_{r_1}(y_1)$$

Taking into account that $y_1y_2y_1^{-1} = y_2z$ and $y_2^{-1}y_1y_2 = zy_1$, we obtain:

$$y_3zy_0 = y_2^{r_2-1}y_1^{r_1-1}+h_{r_2}(y_2)y_1^{-1}+y_2^{-1}y_1^{r_1-1}+y_2^{-1}y_1^{-1} = y_2^{r_2-1}y_1^{r_1-1}+(zy_1)^{r_1-1}y_2^{-1}+y_1^{-1}h_{r_2}(y_2z)+y_1^{-1}z^{-1}y_2^{-1}$$
$$= y_2^{r_2-1}y_1^{r_1-1}+z(y_1z)^{r_1-1}(y_2z)^{-1}+z(y_1z)^{-1}h_{r_2}(y_2z)+z(y_1z)^{-1}(y_2z)^{-1} = y_2^{r_2-1}y_1^{r_1-1}+z(y_1z)^{r_1-1}(y_2z)^{r_2-1}\,.$$

The proposition is proved. □

We expect that the relations in Proposition 10 are defining.

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